ORTHOGONAL POLYNOMIALS WITH RESPECT TO DISCRETE VARIABLES

by

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A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

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KANSAS STATE UNIVERSITY
Manhattan, Kansas

1965

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INTRODUCTION

The functions $\phi(x)$ and $\psi(x)$ are said to be orthogonal to one another on the closed interval $[a,b]$ if

$$\int_a^b \phi(x)\,\psi(x)\,dx = c.$$ [1]

Or if x assumes only discrete values this condition can be written

$$S\left[\phi(x)\,\psi(x)\right] = c,$$

where S denotes summation over the given values x_0, x_1, ... , x_{n-1} of x.

It was Fourier who first used an expansion in a series of orthogonal functions in his treatment of trigonometric series. The first expansion in orthogonal polynomials was preformed by Legendre. In Legendre's polynomials the variable x is a continuous variable on a closed interval $[-1,1]$. Orthogonal polynomials with respect to a discontinuous variable x_0, x_1, ... , x_{n-1} were deduced by Tchebysheff who treated the particular case of two orthogonal polynomials with respect to equidistant variables.[2] R. A. Fisher (1921) was one of the first to use a numerical method of curve fitting by means of orthogonal

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[1] Angus E. Taylor, Advanced Calculus, p. 722

[2] Charles Jordan, "Approximation and Graduation According to the Principle of Least Squares," Annals of Mathematical Statistics, 1932, 3:257

polynomials.

This report will illustrate how a system of orthogonal polynomials with respect to a discrete variable can be obtained and how some methods for obtaining them are derived. It will also show how these polynomials can be very useful in curve fitting.

NUMERICAL METHOD FOR GENERATING ORTHOGONAL POLYNOMIALS

This method utilizes ordinary finite differences (Stanton, 1961). It must therefore be assumed that $x_0, x_1, \ldots, x_{n-1}$ differ by some constant h. They may then be coded so that the values are taken to be 0, 1, ... , n-1.

Using the method of differencing, an equation for P_r, a polynomial of degree r, can be written

$$P_r = \Delta^0 + x\Delta^1 + \frac{x(x-1)}{2!}\Delta^2 + \ldots + \frac{x(x-1)\ldots(x-r+1)}{r!}\Delta^r, \tag{1}$$

where x=0, 1, ... , n-1 and Δ^k represents the k'th ordinary difference of P_r with respect to an initial value, $P_r(0)$.

A statement that P_r is orthogonal to powers of x up to (r-1) can be written

$$S[(x+1)(x+2)\ldots(x+q-1)P_r] = 0, \tag{2}$$

where q=1, 2, ... , r and the sum is over the values of x from 0 to n-1. Now substituting the expression for P_r from equation (1) this condition is

$$S\{[(x+1)(x+2)\ldots(x+q-1)][\Delta^0 + x\Delta^1 + \ldots + \frac{x(x-1)\ldots(x-r+1)}{r!}\Delta^r]\}. \tag{3}$$

If these two terms in the summation of equation (3) are combined and the resulting expression is summed a general term would be

$$S\left[\frac{(x+1)(x+2)\ldots(x+q-1)x(x-1)\ldots(x-p+1)}{p!}\Delta^p\right]. \tag{4}$$

To sum this term multiply numerator and denominator by the quantity $(x+q)-(x-p)$. This then changes expression (4) to

$$S\left\{\left[\frac{\Delta^p}{[(x+q)-(x-p)]p!}\right]\left[(x+q-1)\dots(x+1)x(x-1)\dots(x-p+1)\right]\left[(x+q)-(x-p)\right]\right\}, \tag{5}$$

and multiplying by the quantity $(x+q)-(x-p)$ in the numerator and subtracting the quantity $(x-p)$ from $(x+q)$ in the denominator the expression is

$$S\left\{\frac{\Delta^p}{(q+p)p!}\left[(x+q)(x+q-1)\dots(x+1)x(x-1)\dots(x-p+1)-(x+q-1)\dots(x+1)x(x-1)\dots(x-p)\right]\right\}. \tag{6}$$

The term $\frac{\Delta^p}{(q+p)p!}$ does not contain the index x_i. It may therefore be factored out of the summation leaving

$$\frac{\Delta^p}{(q+p)p!}\left\{S\left[(x+q)(x+q-1)\dots(x+1)x(x-1)\dots(x-p+1)-(x+q-1)\dots(x+1)x(x-1)\dots(x-p)\right]\right\}. \tag{7}$$

Preforming the indicated summation,

$q(q-1)(q-2)\dots(-p+2)(-p+1) \quad - (q-1)(q-2)\dots(-p+1)(-p)$

$(q+1)(q)(q-1)\dots(-p+3)(-p+2) \quad - q(q-1)(q-2)\dots(-p+2)(-p+1)$

$(q+2)(q+1)(q)\dots(-p+4)(-p+3) \quad - (q+1)(q)(q-1)\dots(-p+3)(-p+2)$

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$(n-2+q)(n-3+q)\dots(n-p)(n-1-p) \quad - (n-3+q)(n-4+q)\dots(n-1-p)(n-2-p)$

$(n-1+q)(n-2+q)\dots(n-p+1)(n-p) \quad - (n-2+q)(n-3+q)\dots(n-p)(n-1-p)$

the remaining term is $-(q-1)(q-2)\dots(-p+1)(-p) + (n-1+q)(n-2+q)\dots(n-p+1)(n-p)$ multiplied by $\frac{\Delta^p}{(q+p)p!}$. However, $(q-1)(q-2)\dots(-p+1)(-p)$ must contain a factor of zero. This can be seen by referring to (7) and observing that $(x+q-1)$ is decreased

by one until the quantity (x-p) is obtained. The factor x must therefore be contained in this expression, and with the substitution of x=0 the quantity must also equal zero.

The general term expressed by (4) can now be written

$$\frac{(n-1+q)(n-2+q)\ldots(n-p+1)(n-p)\,\Delta^p}{p!(q+p)}, \tag{8}$$

or as a further simplification,

$$\frac{(n+q-1)!\;\;\Delta^p}{p!\,(n-p-1)!\,(q+p)}. \tag{9}$$

Using this derived general term the condition for orthogonality expressed by (2) or (3) can be replaced by the sum of the general term as p goes from 0 to r. Written algebraically, expression (2) or (3) is equivalent to

$$(n+q-1)!\sum_{p=0}^{r}\frac{\Delta^p}{p!(n-p-1)!(q+p)} = 0, \tag{10}$$

where q=0, 1, ... , r.

It can be seen from equation (10) that the condition of orthogonality is only dependent on Δ^p, all other terms on the right side of equation (10) being positive. If a term can therefore be found for Δ^p which satisfies equation (10), it can be substituted into equation (1) and the resulting expression can be used to generate orthogonal polynomials for successive values of r.

The quantity Δ^p may be found in the following manner. A polynomial function U of degree r may be written

$$U_x = C(x+1)(x+2)\ldots(x+r). \tag{11}$$

Substituting a given value for x the function is

$$U_p = C\frac{(p+r)!}{r!}. \tag{12}$$

Using Lagrange's interpolation formula (Stanton, 1961),

$$U_x = x(x-1)\ldots(x-r)\sum_{p=0}^{r}\frac{(-1)^{r-p}U_p}{p!(r-p)!(x-p)}. \tag{13}$$

With equation (10) in mind x=-q is now substituted in equation (13) which gives

$$U_{-q} = (-1)^{r+1}q(q+1)\ldots(q+r)\sum\frac{(-1)^{r-p}U_p}{p!(r-p)!(-1)(p+q)}. \tag{14}$$

Now if

$$U_p = \frac{(r-p)!(-1)^{r-p}\Delta^p}{(n-p-1)!}, \tag{15}$$

(14) becomes

$$U_{-q} = (-1)^r q(q+1)\ldots(q+r)\sum_{p=0}^{r}\frac{\Delta^p}{p!(n-p-1)!(p+q)}. \tag{16}$$

Referring back to the original equation of U_x (11), it is apparent that U_{-q} is equal to zero for values of q=1, 2, ... , r. The resulting equation is

$$0 = (-1)^r q(q+1)\ldots(q+r)\sum_{p=0}^{r}\frac{\Delta^p}{p!(n-p-1)!(p+q)}. \tag{17}$$

Dividing both sides of equation (17) by the non zero quantity

$(-1)^r q(q+1)\ldots(q+r)$, the equation becomes the condition of orthogonality first derived in (10), ie.,

$$\sum_{p=0}^{r} \frac{\Delta^p}{p!\,(n-p-1)!\,(p+q)} \; . \tag{18}$$

This condition for orthogonality is satisfied when, from equation (15),

$$\Delta^p = \frac{(n-p-1)!\,(-1)^{p-r}\,U_p}{(r-p)!} \; . \tag{19}$$

U_p is given by (12), and in order to hold to the convention of the coefficient of x being unity in a system of orthogonal polynomials, C is set equal to

$$\frac{(r!)^2}{(2r)!\,(n-r-1)!} \; . \tag{20}$$

Rewriting the expression for Δ^p with U_p and C replaced by the stated quantities,

$$\Delta^p = \frac{(n-p-1)!\,(p+r)!}{p!\,(r-p)!} \cdot \frac{(r!)^2\,(-1)^{p-r}}{(n-r-1)!\,(2r)!} \; . \tag{21}$$

Substituting this quantity for Δ^p in (1) and evaluating the terms for successive values of p=0, 1, ... , r, it can be seen that

$$P_r = \frac{(-1)^{-r}(n-1)!\,r!\,(r!)^2}{r!\,(n-r-1)!\,(2r)!} + \frac{(-1)^{1-r}(n-1-1)!\,(1+r)!\,(r!)^2}{1!\,(r-1)!\,(n-r-1)!\,(2r)!}\,X + \tag{22}$$

$$\frac{(-1)^{2-r}(n-2-1)!\,(2+r)!\,(r!)^2}{2!\,(r-2)!\,(n-r-1)!\,(2r)!\,2!}\,X(X-1) + \;.\;.\;.\;.$$

Or expressing P_r in summation notation the general recursion formula is

$$P_r = \frac{(-1)^r (r!)^2 (n-1)!}{(2r)!\,(n-r-1)!} + \tag{23}$$

$$\sum_{p=1}^{r} \left[(-1)^{r-p} \frac{(r!)^2 (p+r)!\,(n-p-1)!}{(2r)!\,(p!)^2 (r-p)!\,(n-r-1)!} \; x(x-1)\ldots(x-p+1) \right].$$

The assumption made at the beginning of this section is also assumed for the recursion formula, namely the x's differ by some constant h.

ALGEBRAIC METHOD FOR GENERATING ORTHOGONAL POLYNOMIALS

In the method presented here no assumptions need be made about the variable x. However, if the assumption of constant differences between the x's is made the arithmetic procedures are greatly reduced.

The expression used for P_r is

$$P_r = \sum_{s=r}^{0} A_s x^s = A_r x^r + A_{r-1} x^{r-1} + \dots + A_0 . \tag{1}$$

where r=0, 1, ... , n-1. The condition of orthogonality considered is

$$S\left[x^k(P_r)\right]=0, \tag{2}$$

where k=0, 1, ... , r-1, and the summation is over all values of x. Substituting (1) into (2) the condition of orthogonality becomes

$$S\left[x^k(A_r x^r + A_{r-1} x^{r-1} + \dots + A_0)\right] = 0 . \tag{3}$$

Now letting k=0, 1, ... , r-1, the set of r equations obtained is

$$\begin{array}{ll} k=0 & S(A_r x^r + A_{r-1} x^{r-1} + \dots + A_0) = 0, \\ k=1 & S(A_r x^{r+1} + A_{r-1} x^r + \dots + A_0 x) = 0, \\ k=2 & S(A_r x^{r+2} + A_{r-1} x^{r+1} + \dots + A_0 x^2) = 0, \\ & \vdots \\ k=r-1 & S(A_r x^{2r-1} + A_{r-1} x^{2r-2} + \dots + A_0 x^{r-1}) = 0. \end{array} \tag{4}$$

Summing the left side of the r equations and dividing both sides by n the set of equations can be expressed as

$$A_r\mu_r + A_{r-1}\mu_{r-1} + \ldots + A_0 = 0,$$
$$A_r\mu_{r+1} + A_{r-1}\mu_r + \ldots + A_0\mu_1 = 0,$$
$$\vdots \qquad\qquad \vdots$$
$$A_r\mu_{2r-1} + A_{r-1}\mu_{2r-2} + \ldots + A_0\mu_{r-1} = 0, \tag{5}$$

where μ_k is the k'th moment about the origin of x. Considering the original equation of P_r (1) and the r equations given by (5), a set of r+1 equations may be written which have r+1 unknowns, A_r, A_{r-1}, ... , A_0. If the assumption is made that $A_r=1$, this set of homogeneous equations is

$$\begin{aligned}(x^r - P_r)A_r + x^{r-1}A_{r-1} + \ldots + A_0 &= 0,\\ \mu_r A_r + \mu_{r-1}A_{r-1} + \ldots + A_0 &= 0,\\ \mu_{r+1}A_r + \mu_r A_{r-1} + \ldots + \mu_1 A_0 &= 0,\\ \vdots \qquad\qquad & \vdots\\ \mu_{2r-1}A_r + \mu_{2r-1}A_{r-1} + \ldots + \mu_{r-1}A_0 &= 0.\end{aligned} \tag{6}$$

For this set of equations to have a non-trivial solution, the determinant of the coefficient matrix must equal zero. Written algebraically, this statement is

$$\begin{vmatrix} (x^r - P_r) & x^{r-1} & x^{r-2} & \ldots & 1 \\ \mu_r & \mu_{r-1} & \mu_{r-2} & \ldots & 1 \\ \mu_{r+1} & \mu_r & \mu_{r-1} & \ldots & \mu_1 \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_{2r-1} & \mu_{2r-2} & \mu_{2r-3} & \ldots & \mu_{r-1} \end{vmatrix} = 0. \tag{7}$$

This statement of equality can be used to obtain the n orthogonal polynomials P_r which satisfy equation (2). Replacing the

given values for r in equation (7) the following results are obtained:

(8)

r=0, $$X^0 - P_0 = 0 \quad \Rightarrow \quad P_0 = 1.$$

r=1, $$\begin{vmatrix} X^1 - P_1 & X^0 \\ \mu_1 & \mu_0 \end{vmatrix} = 0 \quad \Rightarrow \quad P_1 = X - \mu_1 .$$

r=2, $$\begin{vmatrix} X^2 - P_2 & X^1 & X^0 \\ \mu_2 & \mu_1 & \mu_0 \\ \mu_3 & \mu_2 & \mu_1 \end{vmatrix} = 0$$

$$\Rightarrow P_2 = X^2 + \frac{x(\mu_3 - \mu_1\mu_2) + (\mu_2^2 - \mu_1\mu_3)}{(\mu_1^2 - \mu_2)} .$$

r=3, $$\begin{vmatrix} X^3 - P_3 & X^2 & X^1 & X^0 \\ \mu_3 & \mu_2 & \mu_1 & \mu_0 \\ \mu_4 & \mu_3 & \mu_2 & \mu_1 \\ \mu_5 & \mu_4 & \mu_3 & \mu_2 \end{vmatrix} = 0$$

$$\Rightarrow P_3 = X^3 - \frac{X^2(\mu_2^2\mu_3 + \mu_1^2\mu_5 + \mu_3\mu_4 - \mu_2\mu_5 - \mu_1\mu_2\mu_4 - \mu_1\mu_3^2)}{d}$$
$$+ \frac{X(\mu_2\mu_3^2 + \mu_1\mu_2\mu_5 + \mu_4^2 - \mu_3\mu_5 - \mu_2^2\mu_4 - \mu_1\mu_3\mu_4)}{d}$$
$$- \frac{(\mu_3^3 + \mu_2^2\mu_5 + \mu_1\mu_4^2 - \mu_1\mu_3\mu_5 - 2\mu_2\mu_3\mu_4)}{d},$$

where

$$d = \mu_2^3 + \mu_1^2\mu_4 + \mu_3^2 - \mu_2\mu_4 - 2\mu_1\mu_2\mu_3 .$$

The derived expressions for P_r, $r=0, 1, 2, 3$, and the generating equation (7) are true for any spacing of the variable x. If the spacing is constant the equations (7) and (8) can be simplified by subtracting the mean $\overline{x}$ from each variable $x_1, x_2, \ldots, x_n$. This would make μ_k the k'th moment about the mean, $\overline{x}$. The odd moments μ_{2i+1}, $i=0, 1, \ldots$, are then zero. Substituting $\mu_{2i+1}=0$ and $\mu_0=1$ in the expressions for P_0, P_1, P_2, and P_3, the resulting equations are

$$
\begin{aligned}
P_0 &= 1, \\
P_1 &= X, \\
P_2 &= X^2 + (-\mu_2), \\
P_3 &= X^3 + \frac{(\mu_4^2 - \mu_2^2 \mu_4)}{\mu_2^3 - \mu_2 \mu_4} X.
\end{aligned}
\tag{9}
$$

The x's in these equations (9) are the original x's minus $\overline{x}$. P_4 can be obtained by equation (7), but if the μ_{2i+1} are set equal to zero, the equation is simplified to

$$
\begin{vmatrix}
X^4 - P_4 & X^3 & X^2 & X^1 & 1 \\
\mu_4 & 0 & \mu_2 & 0 & 1 \\
0 & \mu_4 & 0 & \mu_2 & 0 \\
\mu_6 & 0 & \mu_4 & 0 & \mu_2 \\
0 & \mu_6 & 0 & \mu_4 & 0
\end{vmatrix} = 0
\tag{10}
$$

$$
\Rightarrow P_4 = X^4 + \frac{x^2(\mu_2^2 \mu_4 \mu_6 + \mu_4^2 \mu_6 - \mu_2 \mu_6^2 - \mu_2 \mu_4^3) + (\mu_2^2 \mu_6^2 + \mu_4^4 - 2\mu_2 \mu_4^2 \mu_6)}{(\mu_2 \mu_4 \mu_6 + \mu_2 \mu_4^2 - \mu_2^3 \mu_6 - \mu_4^3)}.
$$

To obtain a generating function for the case of equal spacing of x the observation is made that

$$
\begin{aligned}
P_0 &= X^0, \\
P_1 &= X^1, \\
P_2 &= X^2 + C_{2,0} X^0, \\
P_3 &= X^3 + C_{3,1} X^1, \\
P_4 &= X^4 + C_{4,2} X^2 + C_{4,0} X^0.
\end{aligned} \tag{11}
$$

Considering the equations expressed by (11) and the determinant in equation (7) with $u_{2i+1} = 0$, a general expression can be written for P . When i is odd

$$P_i = X^i + C_{i,i-2} X^{i-2} + C_{i,i-4} X^{i-4} + \ldots + C_{i,1} X^1, \tag{12}$$

and when i is even

$$P_i = X^i + C_{i,i-2} X^{i-2} + C_{i,i-4} X^{i-4} + \ldots + C_{i,0} X^0. \tag{13}$$

Again considering the equations expressed by (11) it can be seen that

$$
\begin{aligned}
X P_1 &= X^2 = P_2 - C_{2,0} P_0, \\
X P_2 &= X^3 + C_{2,0} X = P_3 - C_{3,1} X + C_{2,0} X = P_3 + (C_{2,0} - C_{3,1}) P_1, \\
X P_3 &= X^4 + C_{3,1} X^2 + C_{4,0} = P_4 + (C_{3,1} - C_{4,2}) P_2 + (C_{2,0} C_{4,2} \\
&\qquad - C_{2,0} C_{3,1} - C_{4,0}) P_0,
\end{aligned} \tag{14}
$$

or

$$X P_i = P_{i+1} + D_{i,1} P_{i-1} + D_{i,2} P_{i-3} + \ldots, \tag{15}$$

where from equation (14)

$$D_{1,1} = -C_{2,0}, \quad D_{2,1} = C_{2,0} - C_{3,1}, \quad D_{3,1} = C_{3,1} - C_{4,2},$$

$$D_{3,2} = C_{2,0}\, C_{4,2} - C_{2,0}\, C_{3,1} - C_{4,0}, \; \ldots .$$

If $D_{3,2}$ were expanded it would be found to equal zero.

Equation (15) expresses $x(P_i)$ as the sum of $\frac{i+3}{2}$ orthogonal polynomials if i is odd, and $\frac{i+2}{2}$ orthogonal polynomials if i is even. To obtain a generating function, (15) is substituted in the expression

$$S(x P_i P_k) = S(P_{i+1} P_k + D_{i,1} P_{i-1} P_k + D_{i,2} P_{i-3} P_k + \ldots). \tag{16}$$

If $i+1 < k$ or $k+1 < i$ this expression will sum to zero because of the orthogonality condition on the P's. However, if successive values of $k = i-3,\ i-5,\ i-7,\ \ldots$ were substituted in the left side of equation (16), and the P's were commuted and summed, a term would be left of the form:

$$S(x P_i P_{i-3}) = S(P_{i+1} P_{i-3} + D_{i,2} P_{i-3}^2 + D_{i,2} P_{i-1} P_{i-3} + \ldots) = D_{i,2}\, S(P_{i-3}^2). \tag{17}$$

$$S(x P_i P_{i-5}) = S(P_{i+1} P_{i-5} + D_{i,1} P_{i-1} P_{i-5} + \ldots + D_{i,3} P_{i-5}^2 + \ldots) = D_{i,3}\, S(P_{i-5}^2).$$

$$\vdots \qquad\qquad \vdots$$

From the fact that the sum of xP_iP_k is equal to zero for one arrangement to P_i and P_k while equal to some quantity $D_{i,g}S(P_q^2)$ when the P's are commuted, it can be inferred that the $D_{i,j}$'s $j>1$ must be zero. If $i=k$ the quantity on the left in expres-

sion (16) would be $S(xP_i^2)$, and examining the right side of the equation it can be seen that this would sum to zero. The only possible combination of i and k left to consider is k=i+1 and k=i-1 for which the expression would be

$$S(xP_iP_{i+1}) = S(P_{i+1}^2 + D_{i,1}P_{i-1}P_{i+1} + \dots) = S(P_{i+1}^2), \tag{18}$$

and

$$S(xP_iP_{i-1}) = S(P_{i+1}P_{i-1} + D_{i,1}P_{i-1}^2 + \dots) = S(D_{i,1}P_{i-1}^2),$$

or

$$S(xP_{i-1}P_i) = S(P_i^2 + D_{i,1}P_{i-2}P_i + \dots) = S(P_i^2).$$

Hence

$$D_{i,1} = \frac{S(xP_iP_{i-1})}{S(P_{i-1}^2)} = \frac{S(xP_{i-1}P_i)}{S(P_{i-1}^2)} = \frac{S(P_i^2)}{S(P_{i-1}^2)}, \tag{19}$$

with all other $D_{i,j}=0$.

Substituting these values of $D_{i,j}$ in equation (15), the equation becomes

$$xP_i = P_{i+1} + \frac{S(P_i^2)}{S(P_{i-1}^2)}P_{i-1} \tag{20}$$

which implies

$$P_{i+1} = xP_i - \frac{S(P_i^2)}{S(P_{i-1}^2)}P_{i-1}. \tag{21}$$

Equation (21) is a recursion formula for P_{i+1} in terms of lower order orthogonal polynomials.

This recursion formula can be used quite easily as it stands. However, if an expression is needed for P_{i+1} in terms of n, $S(P_i^2)$ must be evaluated for the general case. This can be done with the aid of the material from Section One.

From Section One, equation (10), the condition for orthogonality of P_r to a polynomial of degree r-1 is

$$(n+q-1)!\sum_{p=0}^{r}\frac{\Delta^{p}}{p!(n-p-1)!(q+p)}\ ,\quad q=0,1,\ldots,r. \tag{22}$$

If in equation (22) q is set equat to r+1, the results would be the expression for the sum of P_r^2. This can be seen by referring back to equation (2), Section One. Algebraicly the expression would be

$$S(P_r^2)=\sum_{p=0}^{r}\frac{(n+r)!\ \Delta^{p}}{(n-p-1)!\,(p+r+1)}\ . \tag{23}$$

Also, it can be seen from expressions (14), (15), and (16) of Section One that

$$U_{-(r+1)}=C(-r)(-r+1)\ .\ .\ .\ (1)\ , \tag{24}$$

or

$$U_{-(r+1)}=C(-1)^{r}\,r!\ ,$$

and

$$U_{-(r+1)} = (-1)^r (r+1)(r+2)\ldots(2r+1) \sum_{p=0}^{r} \frac{\Delta^p}{p!\,(n-p-1)!\,(p+r+1)} ,$$

or

$$U_{-(r+1)} = (-1)^r \frac{(2r+1)!}{r!} \sum_{p=0}^{r} \frac{\Delta^p}{p!\,(n-p-1)!\,(p+r+1)} ,$$

where -x=q=r+1. Setting the two equations of (24) equal the results are

$$C\,r! = \frac{(2r+1)!}{r!} \sum_{p=0}^{r} \frac{\Delta^p}{p!\,(n-p-1)!\,(p+r+1)} . \tag{25}$$

And solving for

$$\sum_{p=0}^{r} \frac{\Delta^p}{p!\,(n-p-1)!\,(p+r+1)} , \tag{26}$$

it is found equal to

$$\frac{(r!)^2\,C}{(2r+1)!} = \frac{(r!)^4}{(2r)!\,(2r+1)!\,(n-r-1)!} . \tag{27}$$

To obtain the needed expression for $S(P_r^2)$ the expression (26) and (27) are multiplied by (n+r). The resulting expression is

$$S(P_r^2) = \frac{(r!)^4}{(2r)!\,(2r+1)!} \cdot \frac{(n+r)!}{(n-r-1)!} \tag{28}$$

$$= \frac{(r!)^4}{(2r)!\,(2r+1)!}\; n(n^2-1)(n^2-4)(n^2-9)\ldots(n^2-r^2).$$

Hence, from (19),

(29)

$$D_{i,1} = \frac{(n^2-i^2)\,i^2}{4(4i^2-1)}$$

and the recursion formula would be

(30)

$$P_{i+1} = x P_i - \frac{(n^2-i^2)}{4(4i^2-1)}\, P_{i-1}.$$

ORTHOGONAL POLYNOMIALS FOR UNEQUAL INTERVALS

One method was shown for generating orthogonal polynomials for any interval in Section Two. Equation (7) of that section was derived with no restrictions on the independent variable x. It can be seen from that equation that if n is large, even of size 4, the work involved becomes prohibitive. The methods presented in this section will not completely solve this problem. In fact there is no method available at present which treats unequal intervals in which the work is not prohibitive for large n.

Robson (1959) presented a method in which he stated P_r can be constructed recursively from the relation

$$P_r = \frac{1}{c_r}\left(x^r - \sum_{v=0}^{r-1} P_v \sum_{i=1}^{n} x_i^r P_v \right), \tag{1}$$

where c is the normalizing constant

$$c_r^2 = \sum_{j=1}^{n} \left(x^r - \sum_{v=0}^{r-1} P_v \sum_{i=1}^{n} x_i^r P_v \right)^2. \tag{2}$$

The polynomials generated by this method will be orthonormal satisfying the conditions:

$$S(P_i P_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j . \end{cases} \tag{3}$$

Grandage (1958) in an answer to a question by G. W. Snedecor illustrates another method. The method described by Grandage is similiar to the use of equation (7), Section Two. It requires the solution of r linear equations for the construc-

tion of P_r. The method utilized is to construct the succeeding polynomials subject to the constraints of orthogonality. For example, if P_1 is set equal to x+a the given values of x could be substituted into the equation for P_1. The condition $S(P_0P_1)=0$ could then be utilized to solve for a, where $P_0=1$. Similarily in the equation $P_2=x^2+b_2x+a_2$, a_2 and b_2 can be solved for subject to the constraints $S(P_0P_2)=0$ and $S(P_1P_2)=0$.

Wishart and Metakides (1953) illustrates a method for orthogonal polynomial fitting by abbreviated schemes for inverting matrices. The method described in this artical can also be used if there are unequal weights at the different levels. The method is clearly explained in the artical cited and will not be covered in this report.

In the Appendix an example is shown for the derivation of orthogonal polynomials for some artificial data by the three methods presented in this section.

UNDERLYING ASSUMPTIONS IN CURVE FITTING BY ORTHOGONAL POLYNOMIALS

Given n points (x,y) where x is the independent variable and y is the dependent variable it may be desirable to fit a polynomial of degree r to these observations, where r is less than n.

The general regression polynomial that is to be fit is of a general form

$$Y = \sum_{i=0}^{r} A_i P_i , \tag{1}$$

where P_i is a polynomial of degree i subject to the conditions of orthogonality, and A_i is determined by the method of least squares.

The P_i's can be obtained by any of the methods presented in this report with, of course, the restriction on the interval of the x's determining which particular method is used. These methods are simply a way to determine the P_i's so that the following equations are satisfied

$$\begin{aligned} S(P_0\, P_1) &= 0, \\ S(P_0\, P_2) &= 0, \\ S(P_1\, P_2) &= 0, \\ &\vdots \end{aligned} \tag{2}$$

Or in general $S(P_i P_j)=0$ where $i=0, 1, \ldots , j-1$, and S is the summation over all values of x. P_0 is set equal to 1.

To evaluate the coefficients A_i, the method of least squares is utilized, that is, the deviations of y about the regression polynomial are to be minimized by the selection of the A_i's. The sum of the deviations of y about the regression polynomial is expressed by the equation

$$S[\{y - \sum_{i=0}^{r} A_i P_i\}^2] = S[\{y - (A_0 P_0 + A_1 P_1 + \ldots + A_r P_r)\}^2]. \tag{3}$$

The A_i's are now selected to minimize equation (3). Taking repeated derivitives with respect to $A_0, A_1, \ldots , A_r$, the equations obtained are

$$\begin{aligned}
\frac{\partial(3)}{\partial A_0} &= S\{2[y - (A_0 P_0 + A_1 P_1 + \ldots + A_r P_r)(-P_0)]\}, \\
\frac{\partial(3)}{\partial A_1} &= S\{2[y - (A_0 P_0 + A_1 P_1 + \ldots + A_r P_r)(-P_1)]\}, \\
&\vdots \\
\frac{\partial(3)}{\partial A_r} &= S\{2[y - (A_0 P_0 + A_1 P_1 + \cdots + A_r P_r)(-P_r)]\}.
\end{aligned} \tag{4}$$

Multiplying through the right side of (4) by the respective P_i's, the expressions are greatly simplified because of the orthogonality of the P's. Setting the derivitives equal to zero, the equations are

$$S[y P_0 - A_0 P_0^2] = 0, \tag{5}$$

$$S[yP_1 - A_1P_1^2] = 0,$$

$$\vdots \qquad \vdots$$

$$S[yP_r - A_rP_r^2] = 0.$$

Summing the given expressions in (5) and solving for the A's

$$A_0 = \frac{S(yP_0)}{S(P_0^2)}, \tag{6}$$

$$A_1 = \frac{S(yP_1)}{S(P_1^2)},$$

$$\vdots \qquad \vdots$$

$$A_r = \frac{S(yP_r)}{S(P_r^2)}.$$

In testing the goodness of fit of the derived polynomials, the deviations from regression sum of squares is needed. This is obtained from equation (3).

If the expression

$$\left\{ y - \sum_{i=0}^{r} A_i P_i \right\} \tag{7}$$

is squared, the quantity would be

$$y^2 + A_0^2P_0^2 + A_1^2P_1^2 + \ldots + A_r^2P_r^2 - 2[yA_0P_0 + yA_1P_1 + \ldots + yA_rP_r] + 2[A_0A_1P_0P_1 + A_0A_2P_0P_2 + \ldots + A_{r-1}A_rP_{r-1}P_r]. \tag{8}$$

However, when summed, the cross product terms involving P_i, P_j would go to zero because of the orthogonality of P_i and P_j, leaving the term

$$S\{y^2 + \sum_{i=0}^{r} A_i^2 P_i^2 - 2[\sum_{i=0}^{r} y A_i P_i]\}. \tag{9}$$

Utilizing the expression for A_i shown in (6), the quantity becomes

$$S\{y^2 + \frac{S(yP_0)}{S(P_0^2)} A_0 P_0^2 + \frac{S(yP_1)}{S(P_1^2)} A_1 P_1^2 + \ldots + \frac{S(yP_r)}{S(P_r^2)} A_r P_r^2 - 2yA_0P_0 - 2yA_1P_1 - \ldots - 2yA_rP_r\}. \tag{10}$$

Summing over all values of x and factoring the constant values out of the summation, the results are

$$S(y^2) + A_0 \frac{S(yP_0)}{\cancel{S(P_0^2)}} \cancel{S(P_0^2)} + A_1 \frac{S(yP_1)}{\cancel{S(P_1^2)}} \cancel{S(P_1^2)} + \ldots + A_r \frac{S(yP_r)}{\cancel{S(P_r^2)}} \cancel{S(P_r^2)} - 2A_0 S(yP_0) - 2A_1 S(yP_1) - 2A_2 S(yP_2) - \ldots - 2A_r S(yP_r). \tag{11}$$

Combining terms, the sum of squares of the deviations from regression can now be expressed

$$S\{[y - \sum_{i=0}^{r} A_r P_r]^2\} = S(y^2) - \sum_{i=0}^{r} A_i S(P_i y). \tag{12}$$

It can be seen from equation (12) that the reduction in the residual sum of squares is caused by the term A_iP_i, by the quantity $\Sigma A_i S(yP_i)$, or by an equivalent expression, $\Sigma S(A_i^2P_i^2)$.

CONCLUSION

If n points of (x,y) are to be fit by a polynomial of degree r where r is less than n, the method used would be to derive successive polynomials of degree 0, 1, ... , r, stopping where contributions made by the succeeding polynomials are deemed insignificant.* It is realized that if r is not postulated in advance a slight bias is introduced in the estimate of σ^2 (Anderson and Bancroft, 1952). If a method such as least squares is used to determine these successive polynomials all coefficients would have to be determined for each polynomial. For example, if linear regression was first tried, the least squares equation would be

$$y=A+Bx, \tag{1}$$

where the coefficients A and B are determined by the equations

$$\sum_{i=1}^{n} y_i = nA + B\sum_{i=1}^{n} x_i \tag{2}$$

and

$$\sum_{i=1}^{n} x_i y_i = A\sum_{i=1}^{n} x_i + B\sum_{i=1}^{n} x_i^2 .$$

---

*In the case of orthogonal polynomials, by utilizing the partition theorem for the χ^2-distribution (Cochran, 1935) the sum of the deviations about the regression polynomial can be shown to be equal to the sum of Q_1, Q_2, ... , Q_k, variables which are stochastically independent and distributed as χ^2_{k} . A valid 'F' test can then be made on the reduction due to linear regression, then on the additional reduction due to parabolic regression and so on up to the additional reduction due to the regression of the polynomial of degree r-1.

If the deviations from the linear case were found to be significant, a quadratic may be fit to the data. The least squares equation would be

$$y=C+Dx+Ex \ , \tag{3}$$

where the letters C, D and E are used to emphasize the fact that C does not equal A, and D does not equal B. C, D and E can be determined by the equations

$$\sum_{i=1}^{n} y_i = nC + D\sum_{i=1}^{n} x_i + E\sum_{i=1}^{n} x_i^2 , \tag{4}$$

$$\sum_{i=1}^{n} x_i y_i = C\sum_{i=1}^{n} x_i + D\sum_{i=1}^{n} x_i^2 + E\sum_{i=1}^{n} x_i^3 ,$$

$$\sum_{i=1}^{n} x_i^2 y_i = C\sum_{i=1}^{n} x_i^2 + D\sum_{i=1}^{n} x_i^3 + E\sum_{i=1}^{n} x_i^4 .$$

From these equations it can be seen that if r was larger than two, the work involved solving successive regression polynomials could be prohibitive.

The solution of this problem of recalculation all coefficients for each polynomial can be solved by the use of orthogonal polynomials. As stated in this report, the regression polynomial can be written

$$Y = \sum_{i=0}^{r} A_i P_i , \tag{5}$$

where P_i is a polynomial of degree i subject to the conditions of orthogonality and A_i is determined by the method of least squares. The P's can be obtained by the methods outlined in

this report. If the x's are evenly spaced, tables have been constructed for n up to 104 and P_5 (Anderson and Houseman, 1942). The coefficients, A_i, which caused the trouble in other methods can be determined separately in the case of orthogonal polynomials can greatly reduce the work involved in curve fitting.

ACKNOWLEDGMENT

The author would like to express his gratitude to Dr. G. F. Krause, Kansas State University, for his invaluable assistance in preparing this report.

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APPENDIX

Given the set of ordered pairs (0,9), (1,5), (2,6), (4,11), (5,15), (6,17), (8,8) and (10,-1), a polynomial is to be fit to this data. Plotting the points it appears that the desired regression polynomial is a cubic, Fig. (1). The form of the regression polynomial is then

$$Y = \sum_{i=0}^{3} A_i P_i . \tag{1}$$

The first method used to obtain the regression polyno- will be that of Section Three. The equations needed from that section are

$$\begin{aligned} P_0 &= 1, \\ P_1 &= X - \mu_1, \\ P_2 &= X^2 + \frac{X(\mu_3 - \mu_1\mu_2) + (\mu_2^2 - \mu_1\mu_3)}{(\mu_1^2 - \mu_2)}, \\ P_3 &= X^3 - \frac{X^2}{d}(\mu_2^2\mu_3 + \mu_1^2\mu_5 + \mu_3\mu_4 - \mu_2\mu_5 - \mu_1\mu_2\mu_4 - \mu_1\mu_3^2) \\ &\quad + \frac{X}{d}(\mu_2\mu_3^2 + \mu_1\mu_2\mu_5 + \mu_4^2 - \mu_3\mu_5 - \mu_2^2\mu_4 - \mu_1\mu_3\mu_4) \\ &\quad - \frac{1}{d}(\mu_3^3 + \mu_2^2\mu_5 + \mu_1\mu_4 - \mu_1\mu_3\mu_5 - 2\mu_2\mu_3\mu_4), \end{aligned} \tag{2}$$

where

$$d = \mu_2^3 + \mu_1^2\mu_4 + \mu_3^2 - \mu_2\mu_4 - 2\mu_1\mu_2\mu_3 .$$

To obtain the orthogonal polynomials the values of the respec-

tive moments are substituted into the given equations. From the given points the following values are obtained

$$\mu_1 = \frac{\Sigma X}{n} = \frac{36}{8} = \frac{9}{2}, \tag{3}$$

$$\mu_2 = \frac{\Sigma X^2}{n} = \frac{246}{8},$$

$$\mu_3 = \frac{\Sigma X^3}{n} = \frac{1,926}{8},$$

$$\mu_4 = \frac{\Sigma X^4}{n} = \frac{16,290}{8},$$

$$\mu_5 = \frac{\Sigma X^5}{n} = \frac{144,726}{8}.$$

Substituting into the given formulas (1) the equations obtained for the orthogonal polynomials are

$$P_0 = 1, \tag{4}$$

$$P_1 = X - \frac{9}{2},$$

$$P_2 = X^2 + \frac{X\left(\frac{1,906}{8} - \frac{36}{8}\cdot\frac{246}{8}\right) + \left(\frac{246^2}{8^2} - \frac{36}{8}\cdot\frac{1906}{8}\right)}{\frac{36^2}{8^2} - \frac{264}{8}}$$

$$= X^2 + \left(-\frac{78}{8}\right)X + \frac{10.5}{8},$$

$$P_3 = X^3 - \frac{2,129}{141}X^2 + \frac{7,962}{141}X - \frac{4,308}{141}.$$

For ease in further computation P_i is multiplied by λ_i, changing the orthogonal polynomials to the form

$$P_0' = P_0 = 1\,,$$
$$P_1' = 2P_1 = 2x - 9\,,$$
$$P_2' = 8P_2 = 8x^2 - 78x + 105\,,$$
$$P_3' = 141P_3 = 141x^3 - 2{,}129x^2 + 7{,}962x - 4{,}308\,. \tag{5}$$

In solving for the coefficients of the regression polynomial the following table is set up.

Table 1.

x	y	P_0'	P_1'	P_2'	P_3'	yP_0'	yP_1'	yP_2'	yP_3'	
0	9	1	-9	105	-4,308					
1	5	1	-7	35	1,666					
2	6	1	-5	-19	4,228					
4	11	1	-1	-79	2,500					
5	15	1	1	-85	-98					
6	17	1	3	-75	-2,724					
8	8	1	7	-7	-4,676					
10	-1	1	11	125	3,412					
$\Sigma P_i'^2$		8	336	47,376	91,192,904	70	-46	-2,594	-66,172	$\Sigma yP_i'$
λ_i		1	2	8	141					

From Section Five the equation for the coefficients in the regression polynomial is

$$A_i = \frac{S(yP_i)}{S(P_i^2)}\,. \tag{6}$$

The coefficients for this example will therefore be

$$A_0' = \frac{35}{4}\,, \qquad A_2' = -\frac{1{,}297}{23{,}688}\,, \tag{7}$$

$$A_1' = -\frac{23}{168}, \qquad A_3' = -\frac{16,543}{22,798,226}.$$

Substituting the derived quantities, A_i and P_i in equation (1) the desired regression polynomial is

$$Y = \frac{35}{4}(1) + \left(\frac{23}{168}\right)(2x-9) + -\left(\frac{1,297}{23,688}\right)(8x^2-78x+105) \tag{8}$$

$$+ -\left(\frac{16,543}{22,798,226}\right)(141x^3-2,129x^2+7,962x-4,308).$$

If an analysis of variance is now to be preformed on the given data the needed quantities can be obtained from Table (1). As seen in Section Five the residual sum of squares can be expressed

$$S\left\{\left[y - \sum_{i=0}^{r} A_i P_i\right]\right\} = S(y^2) - \sum_{i=0}^{r} A_i S(yP_i).$$

It can be seen from this equation that the reduction in the residual sum of squares is caused by each successive term, $A_iS(yP_i)$. These terms $A_iS(yP_i)$, $i>0$ represent the reduction in the sum of squares caused by linear above the mean, quadratic above linear and so on to the r'th degree above the (r-1)'th degree. For the illustrated example Table (2) shows the respective sum of squares.

Table 2.

Sourse of variation	Degrees of freedom	Sum of squares
Total, Σy^2	7	$Sy^2=842$
Correction for mean	1	$A'_0S(yP'_0)=(35/4)(70)=612.5$
Deviations from mean	6	$842-612.5=229.5$
Linear regression	1	$A'_1S(yP'_1)=(-23/168)(-46)=6.298$
Deviations from linear regression	5	$229.5-6.298=223.202$
Second degree term	1	$A'_2S(yP'_2)=(-1,297/23,688)(2,594)=142.030$
Deviations from quadratic regression	4	$223.202-142.030=81.172$
Third degree term	1	$A'_3S(yP'_3)=(-16,543/22,798)(-66,172)=48.016$
Deviations from third degree regression	3	$81.172-48.016=33.156$

The next method used will be Robson's, Section Four. If equation (1) of Section Four is expanded it can be seen that

$$c_0P_0 = 1, \tag{9}$$

$$c_1P_1 = X - \bar{X},$$

$$c_2P_2 = X^2 - P_0\Sigma X^2P_0 - P_1\Sigma X^2P_1,$$

$$c_3P_3 = X^3 - P_0\Sigma X^3P_0 - P_1\Sigma X^3P_1 - P_2\Sigma X^3P_2.$$

The equations for c_0P_0 and c_1P_1 can be written directly as

$$c_0P_0 = 1, \tag{10}$$

$$c_1P_1 = X - \frac{9}{2},$$

c_2P_2 can be obtained from the table,

x	x^2	c_0P_0	c_1P_1	$x^2c_0P_0$	$x^2c_1P_1$
0	0	1	-9/2		
1	1	1	-7/2		
2	4	1	-5/2		
4	16	1	-1/2		
5	25	1	1/2		
6	36	1	3/2		
8	64	1	7/2		
10	100	1	11/2		
$c_i = \Sigma(c_iP_i)$		$\sqrt{8}$	$\sqrt{336/4}$	246	1,638/2 .

By substituting the given quantities into equation (8),

$$c_2P_2 = X^2 - \frac{(1)}{\sqrt{8}}\left[\frac{246}{\sqrt{8}}\right] - \frac{(X-\frac{9}{2})}{\sqrt{\frac{336}{4}}}\left[\frac{\frac{1638}{2}}{\sqrt{\frac{336}{4}}}\right] \tag{11}$$

$$= X^2 - \frac{78}{8}X + \frac{105}{8} .$$

Setting up a similiar table, c_3P_3 can be obtained.

x^3	c_0P_0	c_1P_1	c_2P_2	$x^3c_0P_0$	$x^3c_1P_1$	$x^3c_2P_2$
0	1	-9/2	105/8			
1	1	-7/2	35/8			
8	1	-5/2	-19/8			
64	1	-1/2	-79/8			
125	1	1/2	-85/8			
216	1	3/2	-75/8			
512	1	7/2	-7/8			
1,000	1	11/2	125/8			
c_i	$\sqrt{8}$	$\sqrt{336/4}$	$\sqrt{47,376/64}$	1926	15,246/2	89,418/8 .

Substituting in the equation (8)

$$c_3P_3 = X^3 - \frac{(1)}{\sqrt{8}}\left[\frac{1,926}{\sqrt{8}}\right] - \frac{(X-\frac{9}{2})}{\sqrt{\frac{336}{4}}}\left[\frac{\frac{15,246}{2}}{\sqrt{\frac{336}{4}}}\right] \tag{12}$$

$$-\frac{\left(x^2-\frac{78}{8}x+\frac{105}{8}\right)}{\sqrt{\frac{47,376}{64}}}\left[\frac{-\frac{89,418}{8}}{\sqrt{\frac{47,376}{64}}}\right]$$

$$= x^3 - \frac{2,129}{141}x^2 + \frac{7,962}{141}x - \frac{4,308}{141}\,.$$

If the c_iP_i's are multiplied by λ_i's the polynomials $\lambda_i c_i P_i$ are identical to those found by the first method shown.

With Robson's orthonormal polynomials

$$A_i = S(yP_i)\,. \tag{13}$$

If however, this formula was changed so that it contained c_iP_i it would be identical to equation (5) and the regression polynomial formed would be equivalent to equation (7).

The last method to be used is that of Grandage. Utilizing the constraints on the orthogonal polynomials that the products must sum to zero the tables are constructed so that on substituting the given values of x and summing the products equal zero.

x	P_0	$P_1=x+a_1$	P_0P_1	$P_1'=2P_1=2x-9$
0	1	a_1		-9
1	1	$1+a_1$		-7
2	1	$2+a_1$		-5
4	1	$4+a_1$		-1
5	1	$5+a_1$		1
6	1	$6+a_1$		3
8	1	$8+a_1$		7
10	1	$10+a_1$		11
			$36+8a_1=0$	

$a_1=-9/2$

It can be seen from the table that a_1 is solved for subject to the constraint of orthogonality. $P_1'=2x-9$, is therefore orthogonal to P_0. In a similiar manner P_2 is obtained.

x	P_0	P_1'	$P_2=x^2+b_2x+a_2$	P_0P_2	$P_1'P_2$
0	1	-9	a_2		
1	1	-7	$1+1b_2+a_2$		
2	1	-5	$4+2b_2+a_2$		
4	1	-1	$16+4b_2+a_2$		
5	1	1	$25+5b_2+a_2$		
6	1	3	$36+6b_2+a_2$		
8	1	7	$64+8b_2+a_2$		
10	1	11	$100+10b_2+a_2$		
				$246+36b_2+8a_2=0$	$1,638+168b_2=0$

$a_2=36(39)-4(246)/4(8)$ $\qquad$ $b_2=-1,638/168$.

Therefore

$$P_2 = x^2 - \frac{39}{4}x + \frac{105}{8}$$

and

$$P_2' = 8P_2 = 8x^2 - 78x + 105.$$

P can be found subject to the constraints

$$S(P_0P_3)=0,$$
$$S(P_1P_3)=0,$$
$$S(P_2P_3)=0,$$

from a similiar table. It is equal to,

$$P_3 = x^3 - \frac{2,129}{141}x^2 + \frac{7,962}{141}x - \frac{4,308}{141} .$$

The orthogonal polynomials are the same as those found

in the first method and therefore the work outlined for obtaining the regression polynomial will be identical.

Figure (1) page 40 shows the original set of points and the derived regression polynomials. The graph indicates how much the succeeding contributions of above the mean, above linear and above quadratic make to the regression polynomial.

If the x's in the preceding example had been evenly spaced the quantities P_1, P_2, and P_3 evaluated at the given x's could have been obtained from Anderson and Houseman (1942). The values of $S(P_i^2)$ are also listed in this reference. If the orthogonal polynomials are needed, P_1 through P_5 are given in this reference and any higher degree can be obtained from the recursion formula, equation (21) or (30) Section Three of this report. The work needed to do the analysis of variance or that of obtaining the regression polynomial would have taken a fraction of the time needed for this example. Given evenly spaced x's Table (1) could have been set up directly. The A_i's would be obtained as shown for the case of the unequal intervals as well as the sum of squares for successive reductions due to each term A_iP_i.

ORTHOGONAL POLYNOMIALS WITH RESPECT TO DISCRETE VARIABLES

by

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AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1965

ABSTRACT

The functions $\varphi(x)$ and $\psi(x)$ are said to be orthogonal to one another on the closed interval $[a,b]$ if

$$\int_a^b \varphi(x)\psi(x)\,dx = 0.$$

Or if x assumes only discrete values this condition can be written

$$S(\varphi(x)\psi(x))=0,$$

where S denotes summation over the given values $x_1, x_2, \ldots, x_n$ of x. This latter condition of orthogonality, orthogonality with respect to discrete variables, is the one discussed in this report.

The derivation of a number of methods for generating orthogonal polynomials for both equal spacing of the variable x and for unequal spacing are shown. Recursion formulas are derived to generate orthogonal polynomials for equal spacing of x.

Given the regression polynomial

$$y = \sum_{i=0}^{r} A_i P_i,$$

where P is a polynomial of degree i subject to the conditions of orthogonality, and A_i is determined by the method of least squares, it is shown that the coefficients can be independently obtained. Utilizing this, numerical methods of curve fitting

can be greatly simplified.

It was the desire of the writer of this report to give the reader both an insight into the derivation of orthogonal polynomials and also some knowledge of their use in curve fitting. The majority of the report deals with the derivation of the orthogonal polynomials with respect to discrete variables.